Multi-Region Two-Group Neutron Diffusion Criticality Calculations with Thermal Feedback for a PWR-W

Project 6, Group 1
NE 571
12/10/2014

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1. Project Introduction

This project expanded upon several previous projects by attempting to loosely couple multiple physics for a typical Westinghouse pressurized water reactor (PWR-W), mimicking the approach utilized by programs such as CASL and NEAMS. In Project 2, the two-group flux distributions and eigenvalues were calculated for a few one-dimensional arrangements of fuel and moderator. In Project 3, the steady state temperature distributions were calculated within a fuel pin, and in Project 5, the time dependent temperature distributions within a fuel pin were calculated based on degraded heat transfer conditions that could lead to meltdown. Using these projects as a base, a multi-region, two group, water-reflected core was constructed for Project 6, and the cross sections were adjusted for thermal feedback. The impact that temperature had upon flux and power distributions, as well as criticality calculations, were then estimated during steady state or transient conditions.

First, the base core design was established by assembling a multi-region two-group variable mesh water-reflected core. The code incorporated three fuel regions and one moderator region (Figure 1), and was able to handle Cartesian, cylindrical, or spherical coordinates. Fuel regions might differ from each other by various combinations of U-235 enrichment, burnable poison loadings, or the presence of control rods. The code was then benchmarked by comparing the numerical value of k-effective to the analytically calculated value. This was achieved by assuming a single region in the fuel and by using the cross sectional data provided in Project 2.

![Figure 1. Schematic of multi-region core](image)

Next, new temperature-dependent cross section libraries were developed for each fuel region in the core. The SCALE code was utilized to generate two group cross section libraries which replaced the cross sectional data used during the benchmark phase. These libraries included values for $\nu \Sigma_f$, $\Sigma_R$, $\Sigma_{S12}$, $\Sigma_a$ and $\Sigma_{tr}$. The temperature-dependent cross sections ranged from those taken at room temperature (20°C) to those taken at the limiting temperature of the
cladding at 1835°C (2108°K) obtained in Project 3. After the cross sections were calculated at various temperatures, a polynomial fit of each cross-section versus fuel temperature was performed, which meant the user did not have to read or interpolate data every time a new cross section was needed. Please note that this step ignored the fact that other parameters in the core may change and thus influence the cross section libraries, such as the moderator density, boron concentration, or burnup of the fuel. Additionally, the temperature dependence of the moderator was ignored. The output for this step included functional fits of the various cross sections as a function of fuel temperature.

Finally, the cross sections were adjusted in the neutronics calculations based on the temperature of the fuel. First, the temperatures in the fueled regions of the core were estimated at steady state operation using thermal calculations similar to those utilized in Projects 3 and 5. Once an average fuel temperature was found, the cross sections were adjusted and the neutronics calculations were repeated until the temperature feedback no longer changed the neutronics results. This entire process was then repeated for three additional core mapping patterns.

By combining the neutronics calculations, temperature-dependent cross sections libraries, and thermal calculations utilized in previous independent projects, a comprehensive multiphysics program was generated for Project 6. Overall, the code was written such that the neutronics module calculated volumetric heat generation data which was then used to determine estimated fuel temperature profiles and effective fuel temperatures. The neutron cross section libraries were then adjusted as a function of these effective fuel temperatures and the neutronics calculations were re-evaluated based on the updated cross sections. This iterative process repeated until the neutronics, temperature distributions, and thermal adjustments to cross section libraries converged. Once this was achieved, the original and final flux maps, fission heat generation, and temperature distributions in each region were plotted and the results were discussed.

2. Background: The Westinghouse PWR

The fuel element modeled is similar to that found in four-loop Westinghouse PWR, with an expected electrical output of 1150 MWe. There are currently 29 operating four-loop Westinghouse PWRs within the United States. The Westinghouse four-loop PWR has 193 fuel

1 (USNRC)
assemblies, each comprised of 17x17 arrays of fuel elements. Figure 2 is an illustration of Westinghouse PWR fuel assembly.

![Figure 2: Typical Fuel Assembly with Rod Cluster Assembly](image)

The fuel pin modeled has the dimensions show in Table 1. The axial dimension is no interest to us as we assume that we can neglect thermal conduction in the axial direction (as the radial temperature gradient across a fuel element is several orders of magnitude greater than the axial direction for PWRs).

<table>
<thead>
<tr>
<th>Diameter Pin</th>
<th>0.94 cm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diameter Fuel</td>
<td>0.8190 cm</td>
</tr>
<tr>
<td>Thickness Gap</td>
<td>0.0082 cm</td>
</tr>
<tr>
<td>Thickness Cladding</td>
<td>0.0572 cm</td>
</tr>
</tbody>
</table>

---

2 (Westinghouse Electric Corporation, 2006)
Figure 3 illustrates a radial view of the fuel element. The fuel pellet consists of UO$_2$ pellets, enriched in U-235 to between 2.10 and 3.10 wt%. The Melting point of the UO$_2$ pellets for this analysis is assumed to be 2847°C (3120°K) and a thermal conductivity of 0.0250 W/cm°K. The fission energy will be assumed to exist as a uniform heat source throughout the fuel. The fuel pellets are contained within a Zircaloy cladding. For this analysis, Zircaloy-1 was used, which is comprised of 2.5wt% Sn, and the remainder as Zr. The melt point of Zircaloy-1 was determined using the metal phase diagram for Zr-Sn$^4$ and estimated to be 1835°C (2108°K) and a thermal conductivity of 0.1070 W/cm°K. Table 2 is a summary of the Westinghouse PWR fuel pellet parameters used as a basis to generate the PWR model.

<table>
<thead>
<tr>
<th>Fuel Pellet Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Material</strong></td>
</tr>
<tr>
<td>Fuel</td>
</tr>
<tr>
<td>UO$_2$</td>
</tr>
<tr>
<td><strong>Enrichment</strong></td>
</tr>
<tr>
<td>2.10% to 3.10%</td>
</tr>
<tr>
<td><strong>Melting Point</strong></td>
</tr>
<tr>
<td>2847°C (3120 K)</td>
</tr>
<tr>
<td><strong>Thermal Conductivity</strong></td>
</tr>
<tr>
<td>0.0250 W/cm°K</td>
</tr>
<tr>
<td><strong>Cladding</strong></td>
</tr>
<tr>
<td>Material</td>
</tr>
<tr>
<td>Zircaloy-1</td>
</tr>
<tr>
<td><strong>Composition</strong></td>
</tr>
<tr>
<td>2.5% Sn, 97.5% Zr</td>
</tr>
<tr>
<td><strong>Melting Point</strong></td>
</tr>
<tr>
<td>1835°C (2108 K)</td>
</tr>
<tr>
<td><strong>Thermal Conductivity</strong></td>
</tr>
<tr>
<td>0.1070 W/cm°K</td>
</tr>
</tbody>
</table>

### Table 2: Westinghouse Fuel Pellet Parameters

#### 3. Methodology

**Step 1: Core Neutronics Model Development**

Core model development was done using MATLAB. Initial core modeling began with the development of a multi-region, two-group, variable mesh water reflected core. The derivations of the finite difference equation solutions to the two-group diffusion equation in each region, at the boundaries and at the interface nodes along with the iterative solution methodology is found in Appendix A. The MATLAB code incorporated the finite difference equations developed in Appendix A in order to provide flux plots for each group and the core k-effective value. Cross-sections used for this portion of model development were default cross section

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3 (Duderstadt & Hamilton, 1976)
4 (Abriata, Sepetember 1983)
used to characterize a general PWR. The MATLAB code developed became the basis for the Core Neutronics Model. Calculations to establish a benchmark were run assuming the core to be a single region. Additionally, a criticality search was performed for a critical radius of 42.5 cm, as seen in Figure 4. As expected, the k-effective approached 1.0.

![Criticality Search](image)

**Figure 4. Criticality search at the critical radius**

**Step 2: Cross Section Module Development**

In order to accurately model the impact of temperature gradients across the core on cross-sections and core reactivity, material libraries were developed for each fuel type used in the core simulator. Once benchmarked, the Core Neutronics Model was expanded to reflect a realistic representation of a PWR by using homogenized 2-group cross sections of a Westinghouse 17x17 assembly. There are 193 fuel assemblies in the core and the square assemblies are placed in the core model as best to approximate a circular shape as shown in Figure 5. The modeled core has a diameter of 337 cm and a fuel region radius of 161.25 cm which accounts for 7.5 assembly pitches. The difference is made up in moderator using cross-sections of water at 316 °C with 600 ppm of boron. Different library types such as burnup, enrichment, control rods in or out are used to approximate actual operating conditions in the core. These library types can be used to model the k-eff, peak power, and leakage of core loading map arrangements and optimize reactor

\(^5\) (Duderstadt & Hamilton, 1976)
operations. Each assembly consists of a 17x17 bundle of fuel pins, instrument tubes, and guide tubes. There are a total of 264 fuel pins, 24 guide tubes, and 1 instrument tube per bundle in the chosen assembly design. The reference assembly design uses a UO$_2$ enrichment of 3.1 wt%.

![Uniform core layout](image)

**Figure 5: Uniform core layout**

**Fuel Cycle Analysis**

Nuclear reactors are operated on a highly controlled schedule to minimize outages and maximize the number of full power days. The number of fresh fuel assemblies loaded during each outage depends on the maximum allowable burnup and the length of the cycle over which the fuel will be depleted. These factors also determine how many cycles an assembly will be kept in the reactor until it has reached the maximum allowable burnup. This calculation was done for the Westinghouse PWR reactor core assuming a power of 3400 MW, a cycle length of 18 months with a capacity factor of 90% and a maximum burnup of 45,000 MWd/MTU. The cycle average burnup is 18,633 MWd/MTU calculated from Equation 1.

$$\text{Cycle Burnup} = \frac{3400 \text{ MW} \times 0.9 \times 18 \times 365 \text{d}}{0.467358 \frac{\text{MW} \cdot \text{d}}{\text{MTU}} \times 193 \text{ Assemblies}}$$  \hspace{1cm} (1)$$

Using the average cycle burnup and the maximum allowable burnup the number cycles an average assembly will be in reactor is found to be 2.415. Because the number of cycles is a fraction between 2 and 3 cycles, 47 assemblies loaded each cycle are burned for 2 cycles and 33 are burned for 3 cycles adding up to 80 fresh assemblies loaded each cycle. From this information the cycle loading scheme shown in Figure 6 shows the two batch loading per cycle.
Cross Section Library Generation

The fuel regions within the core include fresh fuel, once burned fuel, and twice burned fuel to capture the steady-state fissile loading. In order to generate 2-group collapsed cross sections for the 17x17 Westinghouse assembly a depletion code was needed to simulate the reactor history. CASMO4 was chosen for its speed and flexibility to model many different variations of our base-case assembly. For each burnup step a number of branch cases are run to generate collapsed 2-group cross sections for a number of fuel temperatures from room temperature to the fuel melting temperature. The burnup steps chosen are 0, 0.1, 0.5, 1, 5, 10, 15, 20, and 40 GWd/MTU. These cross-sections for each burn up can be grouped into a library and used to generate temperature dependent functions.

Step 3: Core Heat Transfer Model

Model development for heat transfer within the core began with the development of steady state and transient solutions to the heat equation. These solutions are found in Appendix B and C respectively and were developed in earlier projects. For ease of code development, all regional interior nodes were modeled as interface nodes.
Step 4: Coupling Core Neutronics and Heat Transfer for a “Multiphysics” Analysis Tool

The MATLAB model attempts to couple neutronics and thermal dynamics to build a simplified-multiphysics 1-D core simulator. Figure 7 is a flow chart representation of the module logic. As seen in the flowchart, the simulation begins with an initial core geometry and configuration with a variable numbers of regions populated by fuel assemblies with varying degrees of enrichment and burnup. The core regions are further subdivided by the number of assemblies and number of pins according to the size of the region. Each node in the core corresponds to a fuel pin in a radial slice of the core. Furthermore, the terms node and pin can be used interchangeably with respect to the architecture of the code.

![Figure 7: MATLAB Multiphysics Flowchart](image)

The core map is dictated by a vector assigning the desired fuel type to each region of the core where the first value signifies the innermost region of the core and each subsequent value
corresponds to each concentric region from the center of the core to the moderator. The values in the core map vector point to a specific set of cross-sections for a specific fuel type. The cross-section tables generated by SCALE are pre-processed using a polynomial fit function inherent to MATLAB, which calculates the coefficients for a second-order polynomial fit to the each cross-section as a function of temperature. The fitting functions are then used in the main code to calculate temperature-dependent cross-sections for each node and pin in the core.

The core neutronics model uses these generated cross-sections at each node to calculate the flux and subsequently the volumetric heat generation for each node/pin. Two-group diffusion theory is applied to each node separately using the cross-sections corresponding to the specific temperature at each node. The volumetric heat generation, \( q''' \), is required to model the temperature distribution of each fuel pin at each node.

The thermal model can be divided into two subcategories: steady state and transient. Each pin/node has a corresponding temperature profile based on the volumetric heat generation term in that pin. From the pin temperature profile, the effective temperature at each node is then approximated as:

\[
T_{\text{eff}} = \frac{4}{9} T_{\text{CL}} + \frac{5}{9} T_{\text{FS}}
\]  

(2)

where \( T_{\text{CL}} \) is the centerline temperature and \( T_{\text{FS}} \) is the surface temperature. The steady state and transient models return a vector of \( T_{\text{eff}} \) temperatures for each node in the core, which is then passed back to the cross-section interpolation function. Iteration of this process continues until the core thermal profile converges. Steady-state temperatures serve as the initial guess for the transient case. Once the steady-state model is complete, various conditions such as accident scenarios, start-up, and shut down, can be modeled.

**Core Map Module**

Multiple loading patterns were used in the multi-physics code package to show the differences each pattern has on temperature, and flux profiles. These loading maps include the uniform burnup of 32.75 GWd/MTU shown previously in Figure 5 which uses the same cross-section libraries for every node in the fuel region. The “ring of fire” (ROF) loading pattern shown in Figure 8 is an optimized loading pattern that places fresh fuel in a ring and the oldest fuel around the outside. The ROF pattern is used to reduce power peaking as well as neutron leakage.
The next loading pattern simulated is the “out-in” pattern that places all fresh fuel around the edge of the reactor and the burned fuel within the center rings. This pattern flattens power distribution drastically, but has poor neutron economy due to the high leakage out of the reactor. The “out-in” pattern is shown in Figure 9.
A variation of the “out-in” core loading pattern is the “salt-pepper” pattern which still moves the fresh fuel from the outer edge of the core to the center as it is burned. The difference is that the once and twice burned fuels are mixed and checker boarded to spread the reactivity more evenly. This is done in the reactor physics model by giving fractional portions of each assembly width to each type of fuel and alternating between them for each new region. This core design is shown below in Figure 10.

![Figure 10. Salt and Pepper core loading](image)

**4. Results**

**Cross Section Data**

As previously discussed, the cross section libraries for each region were generated using CASMO. Pivotal to the multiphysics approach to core design is the impact of temperature material composition on cross sections. To illustrate this relationship, thermal neutron absorption (Group 2) was plotted as function of temperature and burnup. Figure 11 contains a plot of thermal neutron absorption as a function of temperature. Figure 12 illustrates thermal neutron absorption as a function of temperature.
Figure 11 illustrates that the probability of thermal neutrons being absorbed does not change drastically as the temperature increases. The relationship between neutron absorption and temperature decreases linearly slightly as temperature increases. Burnup has a far more significant impact on the probability of neutron absorption. Figure 12 illustrates an increase in neutron absorption with respect to burnup with a maximum around 15 GWd/kg. The changing
nature of the absorption cross section is due to the changes in isotropic concentrations within the fuel.

Figure 13 and Figure 14 illustrate the impact of neutron generation ($\nu\Sigma_f$) with respect to temperature and burnup for fast flux. Figure 13 illustrates the production of neutrons remains relatively constant as temperature increases.

![Figure 13. $\nu\Sigma_f$ cross sections as a function of temperature](image)

![Figure 14. $\nu\Sigma_f$ cross sections as a function of burnup](image)
In Figure 14, it is clear that as the fuel is burned in the reactor, its ability to produce neutrons decreases across all temperature ranges. This is due to the changing isotopic composition of the fuel. As the reactor operates, concentrations of the fissile material such as U-235 decrease and are replace with other fission products. These isotopes absorb thermal neutrons Figure 11 but contribute less to the generation of neutrons. Over time, the reactivity in the fuel decreases. Understanding the isotopic concentrations and the relationship between neutron absorption and neutron generation aids core designers in balancing reactivity and power distribution within the core so as to produce power equally across the entire region. Equal power production reduces material stresses on supporting material such as the reactor vessel and the core structure.

Next, the normalized neutron flux profile, effective pin temperature profile, fission source heat generation profile, pin centerline and cladding surface temperature profiles, and the peak pin temperature profile were calculated and plotted for each of the four core maps. These values were first found for the uniform core, described in the previous section.

**Uniform Core Map**

The uniform core resembles the multiplying media with reflector simulation performed in project 2 in flux profile shape. This is because only one material is used throughout the fuel region. The heat generation plot shows that towards the edge of the reactor the power rises due to the reflected thermal neutron flux from the moderator region. Overall this loading scheme does not flatten the peak in the center of the core.
Figure 15. Normalized neutron flux profile for “Uniform” slab core

Figure 16. Effective pin temperature profile for “Uniform” slab core
Figure 17. Fission source heat generation profile for “Uniform” slab core

Figure 18. a) Pin centerline and cladding surface temperature profiles for “Uniform” slab core; b) Peak pin temperature profile at r = 0.0 cm for “Uniform” slab core

**Ring of Fire Core Map**

The same calculations were repeated for the “Ring of Fire” core map, as shown in Figure 19 through Figure 22. The main differences in the flux profile is two distinct peak regions which correspond to the once burned and fresh fuel regions which are located in the annulus between the edge and center of the reactor. These two peaks are found in each of the temperature and heat
generation profiles because of the dependence of heat generation on flux. The center of the reactor has twice burned fuel leading to a very low flux compared to the fresh fuel region. This leads to peaking in the fresh fuel region. If this region was more evenly spread out over the core less peaking would occur.

Figure 19. Normalized neutron flux profile for “Ring of Fire” slab core

Figure 20. Effective pin temperature profile for “Ring of Fire” slab core
Figure 21. Fission source heat generation profile for “Ring of Fire” slab core

Figure 22. a) Pin centerline and cladding surface temperature profiles for “Ring of Fire” slab core; b) Peak pin temperature profile at r= 0.192 cm for “Ring of Fire” slab core

**Out-In Core Map**

These calculations were repeated for the “Out-In” core map, as shown in Figure 23 through Figure 26. The out-in core has a ring of fresh fuel around the edge of the reactor leading to a reverse of the uniform loading flux in that the peak occurs at the edge of the reactor versus the center. Because all of the twice burned fuel is lumped into the center region the flux is very low in the center of the core leading to high power peaking and high leakage out of the reactor. This
loading map could be used better if the inner core were mixed with some fresh fuel to even out the peaking seen towards the edge.

Figure 23. Normalized neutron flux profile for “Out-In” slab core

Figure 24. Effective pin temperature profile for “Out-In” slab core
Figure 25. Fission source heat generation profile for “Out-In” slab core

Figure 26. a) Pin centerline and cladding surface temperature profiles for “Out-In” slab core; b) Peak pin temperature profile at r=0.286 cm for “Out-In” slab core

Salt and Pepper Core Map

Finally, these calculations were repeated for the “Salt and Pepper” core map, as shown in Figure 27 through Figure 30. Please notice the discontinuities between the regions in Figure 28 and Figure 30a. The checker boarding done in the center of the reactor core leads to a higher flux than the in-out loading map and a more even distribution of the flux. Because of the fresh fuel
loading around the edge of the reactor the peaking is still quite high for the salt and pepper design.

Figure 27. Normalized neutron flux profile for “Salt and Pepper” slab core

Figure 28. Effective pin temperature profile for “Salt and Pepper” slab core
Figure 29. Fission source heat generation profile for “Salt and Pepper” slab core

Figure 30. a) Pin centerline and cladding surface temperature profiles for “Salt and Pepper” slab core; b) Peak pin temperature profile at r= 0.286 cm for “Salt and Pepper” slab core

5. Discussion

Core design is a delicate balance of power production and temperature generation which requires constant adjustment over the power cycle of the reactor. From the birth of the first neutron within the reactor core, its isotropic concentration changes and the temperature rises. These changes lead to a cascade of every changing variable inputs into the Multiphysics model used to control and monitor core reactivity. Core designs seek to optimize the production of fission energy in order to produce the longest sustained, stable reaction.
This project illustrated the challenges associated with developing cross sections for reactor fuels, which change in composition. It showed that one way to balance the core was to strategically allocated fuel assembly location so as to balance neutron absorption and production producing an equal power distribution. The Ring of Fire core design attempts to do this by surrounded fresh fuel with highly absorbing twice burned fuel. The intent is that the twice-burned fuel absorbs neutrons and minimizes leakage. This design however, results in power peak points where the fresh fuel is loaded. Alternatively, one could design a core like the Out-In core in which all the fresh fuel is on the outside of the core. This design flattens the power distribution across the core but has high leakage and poor neutron economy.

Core design ultimately leads to power distribution which impacts the material life of the tertiary equipment used to support the core. Key components like control rode guide tubes, reactor vessels and coolant pumps are all impacted by the distribution of neutrons, temperature and power across the core. Core designers constantly seek a balance between optimal production of power and the impact on the reactor structure. By understanding how the core changes as the reactor operates, core designs develop the most efficient loading configurations to effectively and safety meet the power production demands of the nuclear energy industry.

One of the issues that arose with this project involved the use of SCALE’s TRITON module. For unknown reasons, the model gave a fission cross section of zero for all cases. Because the problem was never resolved, the group switched to CASMO and temperature-dependent cross sections were generated successfully. Additionally, the steady state and transient thermal models were initially unstable. While the transient model did not converge for any case, the steady state model would converge with a $h_{\infty}$ value greater than 500 W/(m²·K), an unrealistically high value for the heat transfer coefficient. It was determined that this instability was caused by badly condition matrices in the thermal code. After reformatting the matrices, the instability issues were resolved for the steady state model, but due to the time constraints of the project, were not resolved for the transient models. Finally, minor issues arose surrounding the size and complexity of this project. Because multiple independent codes were used in this project, patience and care was needed to ensure that all of the codes compiled correctly and that the proper information was properly passed between them. Overall, the steady state code worked well and produced the expected results.
6. Conclusions

Once again, this project expanded upon several previous projects by attempting to loosely couple multiple physics for a typical PWR-W. Using material properties and dimensions specific to a PWR-W, this code was written such that the volumetric heat generation data calculated in the neutronics module was used to determine the temperature profiles and effective fuel temperatures. The neutron cross section libraries were then adjusted as a function of temperature and the neutronics calculations were re-evaluated based on the updated cross sections. This iterative process repeated until the neutronics, temperature distributions, and thermal adjustments to the cross section libraries converged.

Overall, the multiphysics approach was an accurate and useful introductory model for a PWR-W. The steady state fast and thermal flux, temperature, and heat generation profiles followed the expected shapes for each core map, and the calculated values seemed reasonable. Higher fluxes, temperatures, and heat generation terms were calculated in the core regions with fresh fuel compared to the once-burned and twice burned fuel regions. Additionally, the k-effective values for each steady-state scenario converged to 1.0, as expected. The program worked well and the error was allowably small, thus affirming proper execution of the code. This project was a useful tool which helped us understand the set-up and function of a steady state multiphysics solution.

7. References


Appendix A: Two-Group Diffusion Finite Difference Equation Derivation and Solution

- **Two-Group Diffusion Equations**

  - **Fast Group**  
    \[-\nabla \cdot D_f \nabla \phi_1 + \Sigma_{R_f} \phi_1 = \frac{1}{k} \left[ v_1 \Sigma_{f_1} \phi_1 + v_2 \Sigma_{f_2} \phi_2 \right] \]

  - **Thermal Group**  
    \[-\nabla \cdot D_t \nabla \phi_2 + \Sigma_{a_t} \phi_2 = \Sigma_{s_t} \phi_1 \]

- **Boundary Conditions**
  1. Reflective (mirror) boundary condition: \( \phi^C(r) = \phi^R(r) \)
  2. Reflective (mirror) boundary condition: \( f^C(r) = f^R(r) \)
  3. Zero flux boundary condition: \( \phi^R(r + \Delta r) = 0 \)

- **Left-Most Node for the Core**
  
  - **Fast Group**
    
    \[-D_f r^n \frac{d \phi_1}{dr} \bigg|_0^{\frac{\Delta r}{2}} + \Sigma_{R_f} \phi_1(0) \frac{r^{n+1}}{n+1} \bigg|_0^{\frac{\Delta r}{2}} = \frac{1}{k} v_1 \Sigma_{f_1} \phi_1(0) \frac{r^{n+1}}{n+1} \bigg|_0^{\frac{\Delta r}{2}} + \frac{1}{k} v_2 \Sigma_{f_2} \phi_2(0) \frac{r^{n+1}}{n+1} \bigg|_0^{\frac{\Delta r}{2}} \]
    
    \[-\left( \frac{\Delta r}{2} \right)^n D_f \left[ \phi_1(1) - \phi_1(0) \right] - 0 + \Sigma_{R_f} \phi_1(0) \left[ \frac{(\Delta r)^n}{n+1} - \frac{(0)^{n+1}}{n+1} \right] = \frac{1}{k} v_1 \Sigma_{f_1} \phi_1(0) \left[ \frac{(\Delta r)^n}{n+1} - \frac{(0)^{n+1}}{n+1} \right] + \frac{1}{k} v_2 \Sigma_{f_2} \phi_2(0) \left[ \frac{(\Delta r)^n}{n+1} - \frac{(0)^{n+1}}{n+1} \right] \]

    \[-D_f \left( \frac{\Delta r}{2} \right)^n \left[ \phi_1(1) - \phi_1(0) \right] + \Sigma_{R_f} \phi_1(0) \left[ \frac{(\Delta r)^n}{n+1} \right] = \frac{1}{k} v_1 \Sigma_{f_1} \phi_1(0) \left[ \frac{(\Delta r)^n}{n+1} \right] + \frac{1}{k} v_2 \Sigma_{f_2} \phi_2(0) \left[ \frac{(\Delta r)^n}{n+1} \right] \]

    \[\left[ \frac{D_f}{\Delta r} \left( \frac{\Delta r}{2} \right)^n \right] + \Sigma_{R_f} \left( \frac{\Delta r}{n+1} \right) \phi_1(0) - \frac{D_f}{\Delta r} \left( \frac{\Delta r}{2} \right)^n \phi_1(1) = \frac{1}{k} v_1 \Sigma_{f_1} \phi_1(0) \left[ \frac{(\Delta r)^n}{n+1} \right] + \frac{1}{k} v_2 \Sigma_{f_2} \phi_2(0) \left[ \frac{(\Delta r)^n}{n+1} \right] \]
Therefore, using the boundary condition $-D_1 \frac{\partial \phi_1}{\partial r} |_{r=0} = 0$, the equation for the left-most (or central) node for the fast group in the core is as follows:

$$
\left[ D_1 \left( \frac{\Delta r}{2} \right)^n + \Sigma_{\text{R}_1} \left( \frac{\Delta r \left( \frac{\Delta r}{2} \right)^{n+1}}{n+1} \right) \right] \phi_1(0) - D_1 \left( \frac{\Delta r}{2} \right)^n \phi_1(1) = \frac{1}{k} \nu_1 \Sigma_{\text{f}} \phi_1(0) \left[ \frac{\Delta r \left( \frac{\Delta r}{2} \right)^{n+1}}{n+1} \right] + \frac{1}{k} \nu_2 \Sigma_{\text{f}} \phi_2(0) \left[ \frac{\Delta r \left( \frac{\Delta r}{2} \right)^{n+1}}{n+1} \right]
$$

\[ \text{Central Group} \]

\[ \text{Thermal Group} \]

\[ \text{Central Group} \]

- $\int_0^\frac{\Delta r}{\tau} (-D_2 \nabla^2 \phi_2 + \Sigma_a \phi_2) \, dr = \int_0^\frac{\Delta r}{\tau} (\Sigma_{\text{S}_{12}} \phi_1) \, dr$

- $\int_0^\frac{\Delta r}{\tau} \left( \frac{d}{dr} (\frac{D_2 r^n}{dr}) \frac{\phi_2}{dr} + \Sigma_a \phi_2 \right) \, dr = \int_0^\frac{\Delta r}{\tau} (r^n \Sigma_{\text{S}_{12}} \phi_1) \, dr$

- $\int_0^\frac{\Delta r}{\tau} \left( \frac{d}{dr} (\frac{D_2 r^n}{dr}) \frac{\phi_2}{dr} + \Sigma_a \phi_2 \right) \, dr + \int_0^\frac{\Delta r}{\tau} (\Sigma_{\text{S}_{12}} \phi_2 r^n) \, dr = \int_0^\frac{\Delta r}{\tau} (r^n \Sigma_{\text{S}_{12}} \phi_1) \, dr$

- $-D_2 r^n \frac{d \phi_2}{dr} \bigg|_0^\frac{\Delta r}{\tau} + \Sigma_a \phi_2(0) \frac{r^{n+1}}{n+1} \bigg|_0^\frac{\Delta r}{\tau} = \Sigma_{\text{S}_{12}} \phi_1(0) \frac{r^{n+1}}{n+1} \bigg|_0^\frac{\Delta r}{\tau}$

- $-D_2 \left( \frac{\Delta r}{2} \right)^n \left[ \phi_2(1) - \phi_2(0) \right] - 0 + \Sigma_a \phi_2(0) \left[ \frac{(\Delta r)^{n+1}}{n+1} - \frac{(0)^{n+1}}{n+1} \right] = \Sigma_{\text{S}_{12}} \phi_1(0) \left[ \frac{(\Delta r)^{n+1}}{n+1} - \frac{(0)^{n+1}}{n+1} \right]$

- $-D_2 \left( \frac{\Delta r}{2} \right)^n \left[ \phi_2(1) - \phi_2(0) \right] + \Sigma_a \phi_2(0) \left[ \frac{(\Delta r)^{n+1}}{n+1} \right] = \Sigma_{\text{S}_{12}} \phi_1(0) \left[ \frac{(\Delta r)^{n+1}}{n+1} \right]$

- $\left[ \frac{D_2 \left( \frac{\Delta r}{2} \right)^n}{\Delta r \left( \frac{\Delta r}{2} \right)^{n+1}} + \Sigma_a \left( \frac{(\Delta r)^{n+1}}{n+1} \right) \right] \phi_2(0) - D_2 \left( \frac{\Delta r}{2} \right)^n \phi_2(1) = \Sigma_{\text{S}_{12}} \phi_1(0) \left[ \frac{(\Delta r)^{n+1}}{n+1} \right]$

Therefore, using the boundary condition $-D_2 \frac{\partial \phi_2}{\partial r} |_{r=0} = 0$, the equation for the left-most (or central) node for the thermal group in the core is as follows:

$$
\left[ D_2 \left( \frac{\Delta r}{2} \right)^n + \Sigma_{a_2} \left( \frac{\Delta r \left( \frac{\Delta r}{2} \right)^{n+1}}{n+1} \right) \right] \phi_2(0) - D_2 \left( \frac{\Delta r}{2} \right)^n \phi_2(1) = \Sigma_{\text{S}_{12}} \phi_1(0) \left[ \frac{\Delta r \left( \frac{\Delta r}{2} \right)^{n+1}}{n+1} \right]
$$
• Internal Nodes for the Core

  o Fast Group

  \[
  \int_{r_i - \frac{\Delta r}{2}}^{r_i + \frac{\Delta r}{2}} \left( -D_1 \nabla^2 \phi_1 + \Sigma_{R_1} \phi_1 \right) dr = \int_{r_i - \frac{\Delta r}{2}}^{r_i + \frac{\Delta r}{2}} \left( \frac{1}{k} (v_1 \Sigma_{f_1} \phi_1 + v_2 \Sigma_{f_2} \phi_2) \right) dr
  \]

  \[
  \int_{r_i - \frac{\Delta r}{2}}^{r_i + \frac{\Delta r}{2}} \left( \frac{1}{r^n} \frac{d}{dr} (-D_1 r^n) \frac{d \phi_1}{dr} + \Sigma_{R_1} \phi_1 \right) dr = \int_{r_i - \frac{\Delta r}{2}}^{r_i + \frac{\Delta r}{2}} \left( \frac{1}{k} (v_1 \Sigma_{f_1} \phi_1 + v_2 \Sigma_{f_2} \phi_2) \right) dr
  \]

  \[
  \int_{r_i - \frac{\Delta r}{2}}^{r_i + \frac{\Delta r}{2}} \left( \frac{d}{dr} (-D_1 r^n) \frac{d \phi_1}{dr} + \Sigma_{R_1} \phi_1 r^n \right) dr = \int_{r_i - \frac{\Delta r}{2}}^{r_i + \frac{\Delta r}{2}} \left( \frac{1}{k} (v_1 \Sigma_{f_1} \phi_1 + v_2 \Sigma_{f_2} \phi_2) \right) dr
  \]

  \[
  -D_1 r^n \frac{d \phi_1}{dr} \bigg|_{r_i - \frac{\Delta r}{2}}^{r_i + \frac{\Delta r}{2}} + \Sigma_{R_1} \phi_1 (i) \frac{r^{n+1}}{n+1} \bigg|_{r_i - \frac{\Delta r}{2}}^{r_i + \frac{\Delta r}{2}} = \frac{1}{k} v_1 \Sigma_{f_1} \phi_1 (i) \frac{r^{n+1}}{n+1} \bigg|_{r_i - \frac{\Delta r}{2}}^{r_i + \frac{\Delta r}{2}} + \frac{1}{k} v_2 \Sigma_{f_2} \phi_2 (i) \frac{r^{n+1}}{n+1} \bigg|_{r_i - \frac{\Delta r}{2}}^{r_i + \frac{\Delta r}{2}}
  \]

  Therefore, the equation for an internal node for the fast group in the core is as follows:

  \[
  \left( -\frac{D_1}{\Delta r} \left( r_i - \Delta r \right) \right)^n \phi_1 (i-1) + \left( \frac{D_1}{\Delta r} \left( r_i - \Delta r \right) \right)^n \Sigma_{R_1} \left( \frac{r^n + \Delta r^n}{n+1} - \frac{r_i - \Delta r^n}{n+1} \right) + \frac{D_1}{\Delta r} \left( r_i + \Delta r \right)^n \phi_1 (i) + \left( -\frac{D_1}{\Delta r} \left( r_i + \Delta r \right) \right)^n \phi_1 (i+1)
  \]

  \[
  = \frac{1}{k} v_1 \Sigma_{f_1} \phi_1 (i) \left( \frac{r^n + \Delta r^n}{n+1} - \frac{r_i - \Delta r^n}{n+1} \right) + \frac{1}{k} v_2 \Sigma_{f_2} \phi_2 (i) \left( \frac{r^n + \Delta r^n}{n+1} - \frac{r_i - \Delta r^n}{n+1} \right)
  \]

  o Thermal Group

  \[
  \int_{r_i - \frac{\Delta r}{2}}^{r_i + \frac{\Delta r}{2}} \left( -D_2 \nabla^2 \phi_2 + \Sigma_{a_2} \phi_2 \right) dr = \int_{r_i - \frac{\Delta r}{2}}^{r_i + \frac{\Delta r}{2}} \left( \Sigma_{S_2} \phi_2 \right) dr
  \]

  \[
  \int_{r_i - \frac{\Delta r}{2}}^{r_i + \frac{\Delta r}{2}} \left( \frac{1}{r^n} \frac{d}{dr} (-D_2 r^n) \frac{d \phi_2}{dr} + \Sigma_{a_2} \phi_2 \right) dr = \int_{r_i - \frac{\Delta r}{2}}^{r_i + \frac{\Delta r}{2}} \left( \Sigma_{S_2} \phi_2 \right) dr
  \]
Therefore, the equation for an internal node for the thermal group in the core is as follows:

\[
\left( -\frac{D_i}{\Delta r} \left( r_i - \frac{\Delta r}{2} \right)^n \right) \phi_2(i-1) + \left[ \frac{D_i}{\Delta r} \left( r_i - \frac{\Delta r}{2} \right)^n \phi_2(i) + \Sigma_{a_2} \left( \frac{r_i + \frac{\Delta r}{2}}{n+1} - \frac{r_i - \frac{\Delta r}{2}}{n+1} \right) + \frac{D_i}{\Delta r} \left( r_i + \frac{\Delta r}{2} \right)^n \phi_2(i) \right]
\]

Therefore, the equation for an internal node for the thermal group in the core is as follows:

\[
\left( -\frac{D_i}{\Delta r} \left( r_i - \frac{\Delta r}{2} \right)^n \right) \phi_2(i-1) + \left[ \frac{D_i}{\Delta r} \left( r_i - \frac{\Delta r}{2} \right)^n \phi_2(i) + \Sigma_{a_2} \left( \frac{r_i + \frac{\Delta r}{2}}{n+1} - \frac{r_i - \frac{\Delta r}{2}}{n+1} \right) + \frac{D_i}{\Delta r} \left( r_i + \frac{\Delta r}{2} \right)^n \phi_2(i) \right]
\]

Therefore, the equation for an internal node for the thermal group in the core is as follows:

\[
\left( -\frac{D_i}{\Delta r} \left( r_i - \frac{\Delta r}{2} \right)^n \right) \phi_2(i-1) + \left[ \frac{D_i}{\Delta r} \left( r_i - \frac{\Delta r}{2} \right)^n \phi_2(i) + \Sigma_{a_2} \left( \frac{r_i + \frac{\Delta r}{2}}{n+1} - \frac{r_i - \frac{\Delta r}{2}}{n+1} \right) + \frac{D_i}{\Delta r} \left( r_i + \frac{\Delta r}{2} \right)^n \phi_2(i) \right]
\]

**Core-Reflector Interface**

- Fast Group

\[\int_{r_1}^{r_i} \frac{d}{dr} \left( -D_1 \nabla^2 \phi_1 + \Sigma_{R_1} \phi_1 \right) dr + \int_{r_i}^{r_{i+1}} \frac{d}{dr} \left( -D_1 \nabla^2 \phi_1 + \Sigma_{R_1} \phi_1 \right) dr = \int_{r_1}^{r_i} \frac{1}{k} \left( \nu_1 \Sigma_{f_1} \phi_1 + \nu_2 \Sigma_{f_2} \phi_2 \right) dr \]

\[= \frac{1}{k} \left( \nu_1 \Sigma_{f_1} \phi_1 + \nu_2 \Sigma_{f_2} \phi_2 \right) \frac{r_{i+1} - r_{i-1}}{n+1} \]

\[= \frac{1}{k} \left( \nu_1 \Sigma_{f_1} \phi_1 + \nu_2 \Sigma_{f_2} \phi_2 \right) \frac{r_{i+1} - r_{i-1}}{n+1} \]

\[= \frac{1}{k} \left( \nu_1 \Sigma_{f_1} \phi_1 + \nu_2 \Sigma_{f_2} \phi_2 \right) \frac{r_{i+1} - r_{i-1}}{n+1} \]

\[= \frac{1}{k} \left( \nu_1 \Sigma_{f_1} \phi_1 + \nu_2 \Sigma_{f_2} \phi_2 \right) \frac{r_{i+1} - r_{i-1}}{n+1} \]

\[= \frac{1}{k} \left( \nu_1 \Sigma_{f_1} \phi_1 + \nu_2 \Sigma_{f_2} \phi_2 \right) \frac{r_{i+1} - r_{i-1}}{n+1} \]

\[= \frac{1}{k} \left( \nu_1 \Sigma_{f_1} \phi_1 + \nu_2 \Sigma_{f_2} \phi_2 \right) \frac{r_{i+1} - r_{i-1}}{n+1} \]

\[= \frac{1}{k} \left( \nu_1 \Sigma_{f_1} \phi_1 + \nu_2 \Sigma_{f_2} \phi_2 \right) \frac{r_{i+1} - r_{i-1}}{n+1} \]

\[= \frac{1}{k} \left( \nu_1 \Sigma_{f_1} \phi_1 + \nu_2 \Sigma_{f_2} \phi_2 \right) \frac{r_{i+1} - r_{i-1}}{n+1} \]

\[= \frac{1}{k} \left( \nu_1 \Sigma_{f_1} \phi_1 + \nu_2 \Sigma_{f_2} \phi_2 \right) \frac{r_{i+1} - r_{i-1}}{n+1} \]

\[= \frac{1}{k} \left( \nu_1 \Sigma_{f_1} \phi_1 + \nu_2 \Sigma_{f_2} \phi_2 \right) \frac{r_{i+1} - r_{i-1}}{n+1} \]

\[= \frac{1}{k} \left( \nu_1 \Sigma_{f_1} \phi_1 + \nu_2 \Sigma_{f_2} \phi_2 \right) \frac{r_{i+1} - r_{i-1}}{n+1} \]

\[= \frac{1}{k} \left( \nu_1 \Sigma_{f_1} \phi_1 + \nu_2 \Sigma_{f_2} \phi_2 \right) \frac{r_{i+1} - r_{i-1}}{n+1} \]

\[= \frac{1}{k} \left( \nu_1 \Sigma_{f_1} \phi_1 + \nu_2 \Sigma_{f_2} \phi_2 \right) \frac{r_{i+1} - r_{i-1}}{n+1} \]

\[= \frac{1}{k} \left( \nu_1 \Sigma_{f_1} \phi_1 + \nu_2 \Sigma_{f_2} \phi_2 \right) \frac{r_{i+1} - r_{i-1}}{n+1} \]

\[= \frac{1}{k} \left( \nu_1 \Sigma_{f_1} \phi_1 + \nu_2 \Sigma_{f_2} \phi_2 \right) \frac{r_{i+1} - r_{i-1}}{n+1} \]

\[= \frac{1}{k} \left( \nu_1 \Sigma_{f_1} \phi_1 + \nu_2 \Sigma_{f_2} \phi_2 \right) \frac{r_{i+1} - r_{i-1}}{n+1} \]

\[= \frac{1}{k} \left( \nu_1 \Sigma_{f_1} \phi_1 + \nu_2 \Sigma_{f_2} \phi_2 \right) \frac{r_{i+1} - r_{i-1}}{n+1} \]
The finite-difference equation for the interface node between the core and the reflector for the fast group is as follows:

\[
\begin{align*}
(r_i - \frac{\Delta r}{2}) D_r \phi_i(i) &+ \sum_{\beta_i} \phi_i(i) \left[ \frac{(r_{i})^{n+1} - (r_i - \frac{\Delta r}{2})}{n+1} \right] - \left( r_i + \frac{\Delta r}{2} \right) D_r \phi_i(i) \\
&= \frac{1}{k} \nu_r \Sigma_f \phi_i(i) \left[ \frac{(r_{i})^{n+1} - (r_i - \frac{\Delta r}{2})}{n+1} \right] + \frac{1}{k} \nu_r \Sigma_f \phi_2(i) \left[ \frac{(r_{i})^{n+1} - (r_i - \frac{\Delta r}{2})}{n+1} \right]
\end{align*}
\]

○ Thermal Group

\[
\int_{r_i - \frac{\Delta r}{2}}^{r_i} (-D_2 \nabla^2 \phi_2 + \Sigma_{a_2} \phi_2) dr + \int_{r_i - \frac{\Delta r}{2}}^{r_i + \frac{\Delta r}{2}} (-D_2 \nabla^2 \phi_2 + \Sigma_{a_2} \phi_2) dr = \int_{r_i - \frac{\Delta r}{2}}^{r_i} \left( \Sigma_{s_{12}} \phi_1 \right) dr + \int_{r_i - \frac{\Delta r}{2}}^{r_i + \frac{\Delta r}{2}} \left( \Sigma_{s_{12}} \phi_1 \right) dr
\]

\[
\begin{align*}
-D_2 \frac{d^2 \phi_2}{d r^2} \bigg|_{r_i - \frac{\Delta r}{2}}^{r_i} + \Sigma_{a_2} \phi_2(i) \left[ \frac{r_{i}^{n+1} - (r_i - \frac{\Delta r}{2})}{n+1} \right] + \Sigma_{a_2} \phi_2(i) \left[ \frac{r_{i}^{n+1} - (r_i - \frac{\Delta r}{2})}{n+1} \right] = \Sigma_{s_{12}} \phi_1(i) \left[ \frac{r_{i}^{n+1} - (r_i - \frac{\Delta r}{2})}{n+1} \right] + \Sigma_{s_{12}} \phi_1(i) \left[ \frac{r_{i}^{n+1} - (r_i - \frac{\Delta r}{2})}{n+1} \right]
\end{align*}
\]
The finite-difference equation for the interface node between the core and the reflector for the thermal group is as follows:

\[
\begin{align*}
(r_i - \frac{\Delta r}{2})^n D_2 \left[ \frac{\phi_2(i) - \phi_2(i-1)}{\Delta r} \right] + \Sigma_{a_2} \phi_2(i) \left[ \frac{(r_i)^{n+1} - (r_i - \frac{\Delta r}{2})^{n+1}}{n+1} \right] & - (r_i + \frac{\Delta r}{2})^n D_2 \frac{\phi_2(i+1) - \phi_2(i)}{\Delta r} \\
+ \Sigma_{a_2} R \phi_2(i) \left[ \frac{(r_i + \frac{\Delta r}{2})^{n+1} - (r_i)^{n+1}}{n+1} \right] & = \Sigma_{s_{12}} \phi_1(i) \left[ \frac{(r_i)^{n+1} - (r_i - \frac{\Delta r}{2})^{n+1}}{n+1} \right] + \Sigma_{s_{12}} R \phi_1(i) \left[ \frac{(r_i + \frac{\Delta r}{2})^{n+1} - (r_i)^{n+1}}{n+1} \right]
\end{align*}
\]

The finite-difference equation for the interface node between the core and the reflector for the thermal group is as follows:

\[
\begin{align*}
- (r_i - \frac{\Delta r}{2})^n D_2 \frac{\phi_2(i-1)}{\Delta r} & + \left( r_i - \frac{\Delta r}{2} \right)^n D_2 + \Sigma_{a_2} \left[ (r_i)^{n+1} - (r_i - \frac{\Delta r}{2})^{n+1} \right] \frac{r_i + \frac{\Delta r}{2}}{\Delta r} D_2 \frac{R}{\Delta r} \\
+ \Sigma_{a_2} R \left[ \frac{(r_i + \frac{\Delta r}{2})^{n+1} - (r_i)^{n+1}}{n+1} \right] & = \Sigma_{s_{12}} \phi_1(i) \left[ \frac{(r_i)^{n+1} - (r_i - \frac{\Delta r}{2})^{n+1}}{n+1} \right] + \Sigma_{s_{12}} R \phi_1(i) \left[ \frac{(r_i + \frac{\Delta r}{2})^{n+1} - (r_i)^{n+1}}{n+1} \right]
\end{align*}
\]

- **Internal Nodes for the Reflector**

  - **Fast Group**

\[
\begin{align*}
& \int_{r_i - \frac{\Delta r}{2}}^{r_i + \frac{\Delta r}{2}} \left( -D_1 R \nabla^2 \phi_1 + \Sigma_{R_1} R \phi_1 \right) dr = 0 \\
& \int_{r_i - \frac{\Delta r}{2}}^{r_i + \frac{\Delta r}{2}} \left( \frac{1}{r^n} \frac{d}{dr} \left( -D_1 R r^n \right) \frac{d\phi_1}{dr} + \Sigma_{R_1} R \phi_1 \right) dr = 0 \\
& \int_{r_i - \frac{\Delta r}{2}}^{r_i + \frac{\Delta r}{2}} \left( \frac{d}{dr} \left( -D_1 R r^n \phi_1 r^n \right) + \Sigma_{R_1} R \phi_1 r^n \right) dr = 0 \\
& \int_{r_i - \frac{\Delta r}{2}}^{r_i + \frac{\Delta r}{2}} \left( \frac{d}{dr} \left( -D_1 R r^n \phi_1 r^n \right) + \Sigma_{R_1} R \phi_1 r^n \right) dr = 0
\end{align*}
\]
Therefore, the equation for an internal node for the fast group in the reflector is as follows:

\[-D_1^R r^n \frac{d\phi_1}{dr} \bigg|_{r_i-\frac{\Delta r^R}{2}}^{r_{i+1}^+} + \sum_{R_1} \frac{\phi_1(i) r^{n+1}}{n+1} \bigg|_{r_i-\frac{\Delta r^R}{2}}^{r_{i+1}^+} = 0\]

\[-\left(\eta_i + \frac{\Delta r^R}{2}\right)^n D_1^R \left[\phi_1(i+1) - \phi_1(i)\right] + \left(\eta_i - \frac{\Delta r^R}{2}\right)^n D_1^R \left[\phi_1(i) - \phi_1(i-1)\right] + \sum_{R_1} \frac{\phi_1(i)}{n+1} \left[\frac{\eta_i + \frac{\Delta r^R}{2}}{n+1} - \frac{\eta_i - \frac{\Delta r^R}{2}}{n+1}\right] = 0\]

Therefore, the equation for an internal node for the thermal group in the reflector is as follows:

\[\left(-\frac{D_1^R}{\Delta r^R} \left(\eta_i - \frac{\Delta r^R}{2}\right)^n\right) \phi_1(i-1) + \left[D_1^R \left[\eta_i + \frac{\Delta r^R}{2}\right] + \sum_{R_1} \frac{\phi_1(i) - \phi_1(i-1) + \phi_1(i+1)}{n+1}\right] + \left(-\frac{D_1^R}{\Delta r^R} \left(\eta_i - \frac{\Delta r^R}{2}\right)^n\right) \phi_1(i) = 0\]

### Thermal Group

\[
\int_{r_i-\Delta r^R}^{r_{i+1}^+} \left(-D_2^R r^n \frac{d\phi_2}{dr} + \sum_{a_2} \frac{\phi_2}{n+1}\right) dr = \int_{r_i-\Delta r^R}^{r_{i+1}^+} \left(\sum_{s_1} \frac{\phi_1}{n+1}\right) dr
\]

\[
\int_{r_i-\Delta r^R}^{r_{i+1}^+} \left(-D_2^R r^n \frac{d\phi_2}{dr} + \sum_{a_2} \frac{\phi_2 r^n}{n+1}\right) dr = \int_{r_i-\Delta r^R}^{r_{i+1}^+} \left(\sum_{s_1} \frac{\phi_1 r^n}{n+1}\right) dr
\]

\[
\int_{r_i-\Delta r^R}^{r_{i+1}^+} \left(\frac{d}{dr}(-D_2^R r^n) \frac{d\phi_2}{dr} + \sum_{a_2} \frac{\phi_2 r^n}{n+1}\right) dr = \int_{r_i-\Delta r^R}^{r_{i+1}^+} \left(\sum_{s_1} \frac{\phi_1 r^n}{n+1}\right) dr
\]

\[
(-D_2^R r^n) \frac{d\phi_2}{dr} \bigg|_{r_{i-\Delta r^R}}^{r_{i+1}^+} + \sum_{a_2} \frac{\phi_2(i) r^{n+1}}{n+1} \bigg|_{r_{i-\Delta r^R}}^{r_{i+1}^+} = \sum_{s_1} \frac{\phi_1(i) r^{n+1}}{n+1} \bigg|_{r_{i-\Delta r^R}}^{r_{i+1}^+}
\]

\[-\left(\eta_i + \frac{\Delta r^R}{2}\right)^n D_2^R \left[\phi_2(i+1) - \phi_2(i)\right] + \left(\eta_i - \frac{\Delta r^R}{2}\right)^n D_2^R \left[\phi_2(i) - \phi_2(i-1)\right] + \sum_{a_2} \frac{\phi_2(i)}{n+1} \left[\frac{\eta_i + \frac{\Delta r^R}{2}}{n+1} - \frac{\eta_i - \frac{\Delta r^R}{2}}{n+1}\right] = \sum_{s_1} \frac{\phi_1(i) r^{n+1}}{n+1} \bigg|_{r_{i-\Delta r^R}}^{r_{i+1}^+} - \left(\eta_i - \frac{\Delta r^R}{2}\right)^n D_2^R \left[\phi_2(i+1) - \phi_2(i)\right] + \left(\eta_i + \frac{\Delta r^R}{2}\right)^n D_2^R \left[\phi_2(i) - \phi_2(i-1)\right] + \sum_{a_2} \frac{\phi_2(i)}{n+1} \left[\frac{\eta_i + \frac{\Delta r^R}{2}}{n+1} - \frac{\eta_i - \frac{\Delta r^R}{2}}{n+1}\right] = 0
\]

Therefore, the equation for an internal node for the thermal group in the reflector is as follows:
\[
\left( -\frac{D_2^R}{\Delta r^R} \left( r_i - \frac{\Delta r^R}{2} \right) \right)^n \phi_2(i-1) + \left[ \frac{D_2^R}{\Delta r^R} \left( r_i - \frac{\Delta r^R}{2} \right) \right]^n + \sum_{a_2} \left( \frac{r_i + \Delta r^R}{2} \right)^{n+1} - \left( \frac{r_i - \Delta r^R}{2} \right)^{n+1} + \left( -\frac{D_2^R}{\Delta r^R} \left( r_i + \frac{\Delta r^R}{2} \right) \right)^n \phi_2(i) + \left( -\frac{D_2^R}{\Delta r^R} \left( r_i + \frac{\Delta r^R}{2} \right) \right)^n \phi_2(i+1) = \sum_{s_{12}} \phi_1(i) \left[ \frac{r_i + \Delta r^R}{2} \right]^{n+1} - \left( \frac{r_i - \Delta r^R}{2} \right)^{n+1} \]

- **Right-Most Node for the Reflector**
  \[
  \phi_1(r + \Delta r) = 0 \\
  \phi_2(r + \Delta r) = 0
  \]

- **Solution Approach**
  1. Guess \( S \)
  2. Solve for \( \phi_1: \quad [A_1] [\phi_1] = \frac{1}{k} [S] \)
  3. Solve for \( \phi_2: \quad [A_2] [\phi_2] = [B] [\phi_1] \)
  4. Solve for \( S \) and \( k: \quad [S] = [C_1] [\phi_1] + [C_2] [\phi_2] \)
  5. Repeat the process with new \( S \) and \( k \):
      \[ k_{new} = k_{old} \frac{s_{new}}{s_{old}} \]

\( A_1 = \) L.H.S. of fast group equation
\( A_2 = \) L.H.S. of thermal group equation
\( B = \) R.H.S. of thermal group equation (Down Scattering)
\( C_1 = \) R.H.S. of fast group equation for 1 (Fast Flux)
\( C_2 = \) R.H.S. of fast group equation for 2 (Thermal Flux)
Appendix B: Analytical Steady State Derivations of the Heat Transfer Equation

Fuel Temperature Derivation:

\[
K \frac{1}{r} \frac{d}{dr} \left( r \frac{dT}{dr} \right) = -q'''
\]

\[
K \frac{d}{dr} \left( r \frac{dT}{dr} \right) = -q'''r
\]

\[
\int_0^r K_f \frac{d}{dr} \left( r \frac{dT}{dr} \right) dr = \int_0^r -q'''r dr
\]

\[
K_f r \frac{dT}{dr} = -q''r^2
\]

\[
\frac{dT}{dr} = \frac{-q''r}{2K_f}
\]

\[
T(r) = T_{cl} - \frac{q'' r^2}{4K_f}
\]

Gas Gap Temperature Derivation:

\[
\frac{1}{r} \frac{d}{dr} \left( K_g r \frac{dT}{dr} \right) = 0
\]

\[
\frac{d}{dr} \left( K_g r \frac{dT}{dr} \right) = 0
\]

\[
\int_{r_f}^{r_g} \frac{d}{dr} \left( K_g r \frac{dT}{dr} \right) dr = 0
\]

\[
K_g r \frac{dT}{dr} = K_f r_f \frac{dT}{dr} = q'' = \frac{q'''}{2}
\]

\[
q'' = \frac{\pi r_f^2}{2}
\]

\[
\frac{dT}{dr} = \frac{q'''}{2K_g r}
\]

\[
T(r) = T(r_f) - \frac{q'''}{2K_g r} \ln \left( \frac{r_f}{r} \right)
\]

Cladding Temperature Derivation:

\[
\frac{1}{r} \frac{d}{dr} \left( K_c r \frac{dT}{dr} \right) = 0
\]

\[
\frac{d}{dr} \left( K_c r \frac{dT}{dr} \right) = 0
\]

\[
\int_{r_f}^{r_g} \frac{d}{dr} \left( K_c r \frac{dT}{dr} \right) dr = 0
\]

\[
\int_{r_f}^{r_g} \frac{d}{dr} \left( K_c r \frac{dT}{dr} \right) dr = 0
\]
\[ K_c r \frac{dT}{dr} - K_g r_g \frac{dT}{dr} = 0 \]

\[ K_c r \frac{dT}{dr} = K_g r_g \frac{dT}{dr} = q'' = \frac{q'''}{2} r_f \]

\[ \frac{q''}{q'''} = \frac{\pi r_f^2}{2 \pi r_f} = \frac{r_f}{2} \]

\[ \frac{dT}{dr} = \frac{q'''}{2} r_f^2 \]

\[ T(r) = T(r_g) - \frac{q'''}{2K_c r} \ln \left( \frac{r_f + r_g}{r} \right) \]
Implicit Solution Method

The alternative solution to the explicit method is the implicit method. It generates solutions that are unconditionally stable. The solution must be solved using a system of linear equations. This solution method might result in longer computational times depending on the size of the system. The development of an implicit solution begins with the derivations of the linear equations using finite difference methods.

The implicit solution of the heat equation for a one-dimensional slab begins with Eq. D1.

\[
\frac{d}{dt}(\rho c T) - \nabla k \nabla T = q'''(r, t) \quad \text{(Eq. D1)}
\]

Recall that \( \nabla k \nabla T \) is \( k \nabla^2 T \) where

\[
\nabla^2 = \frac{1}{r^n} \frac{d}{dr} r^n \frac{d}{dr} \quad \text{(Eq. D2)}
\]

For a 1D cylinder, \( n=1 \) and thus Eq. D1 becomes

\[
\frac{d}{dt}(\rho c T) = k_f \frac{1}{r} \frac{d}{dr} r \frac{d}{dr} + q'''
\]

(Multiplying \( r \) through the equation results in the form that will be used for the basis of the following derivations.

\[
\frac{d}{dt}(\rho c T r) = k_f \frac{d}{dr} r \frac{d}{dr} + q''' r \quad \text{(Eq. D4)}
\]

Eq. D4 is then used to generate finite difference equations for nodes within the cylinder. The following paragraphs derive the solutions for the finite difference equations at key locations within the cylinder.

The following boundary conditions apply to this problem.

**BC1:** \( q'' = -k_f \left. \frac{dT}{dr} \right|_0 = 0 \)

**BC2:** \( q''_f = q''_g = k_f r_f \left. \frac{dT}{dr} \right|_{r_f} = k_g r_f \left. \frac{dT}{dr} \right|_{r_f} \)

**BC3:** \( q''_g = q''_c = k_g r_i \left. \frac{dT}{dr} \right|_{r_i} = k_c r_i \left. \frac{dT}{dr} \right|_{r_i} \)

**BC4:** \( -k_c \left. \frac{dT}{dt} \right|_R = h_{\infty} (T_R - T_{\infty}) \)
Derivation Finite Difference Equation: LHS Node (Center)

The implicit solution for the left most node begins with Eq. D4 and the boundary condition illustrated in Figure D-1. Integration is conducted over time from \( t_p \) to \( t_{p+1} \) and over space from \( \frac{\Delta r_f}{2} \) to 0.

\[
\int_{t_p}^{t_{p+1}} \int_0^{\Delta r_f} \left( \frac{d}{dt} (\rho c T_r) \right) dr dt = \int_{t_p}^{t_{p+1}} \int_0^{\Delta r_f} \left( k_f \frac{d}{dr} T_r \right) dr dt + \int_{t_p}^{t_{p+1}} \int_0^{\Delta r_f} q'' r dr dt \quad \text{(Eq. D5)}
\]

Solving for the left side of Eq. D5 results in the following

\[
\int_{t_p}^{t_{p+1}} \int_0^{\Delta r_f} \left( \frac{d}{dt} (\rho c T_r) \right) dr dt = \int_{t_p}^{t_{p+1}} \frac{\rho c d}{dt} \left( \frac{\Delta r_f}{2} \right)^2 dt \quad \text{(Eq. D6)}
\]

\[
\int_{t_p}^{t_{p+1}} \frac{\rho c d}{dt} \left( \frac{\Delta r_f}{2} \right)^2 dt = \frac{\rho c T_i^p \Delta r_f^2}{8} - \frac{\rho c T_{i+1}^p \Delta r_f^2}{8} \quad \text{(Eq. D7)}
\]

When solving for the right side of Eq. D5, \( t_{p+1} = t_p + \Delta t \). Solving for the right side of Eq. D5 results in the following

\[
\int_{t_p}^{t_{p+1}} \int_0^{\Delta r_f} \left( k_f \frac{d}{dr} T_r \right) dr dt + \int_{t_p}^{t_{p+1}} \int_0^{\Delta r_f} q'' r dr dt = \int_{t_p}^{t_{p+1}} k_f \frac{d}{dr} \left( \frac{T_i+1 - T_i}{\Delta r_f} \right) dt + \int_{t_p}^{t_{p+1}} \frac{q'' \Delta r_f}{2} \left( \frac{\Delta r_f}{2} \right)^2 dt \quad \text{(Eq. D8)}
\]

Applying BC1, \(-k_f \frac{dT_i}{dr} = 0\), to Eq. D9 results in the following simplification.

\[
\left[ \frac{k_f}{\Delta r_f} \left( \frac{\Delta r_f}{2} \right) T_{i+1}^p + \frac{k_f}{\Delta r_f} \left( \frac{\Delta r_f}{2} \right) T_i^p \right] \Delta t + \frac{q'' \Delta r_f^2 \Delta t}{8} \quad \text{(Eq. D10)}
\]

Combining Eq. D7 and Eq. D10 results in the full finite difference equation for the left most node.

\[
\frac{\rho c T_i^p \Delta r_f^2}{8} - \frac{\rho c T_{i+1}^p \Delta r_f^2}{8} = k_f \frac{\Delta t}{2} T_i^{p+1} - k_f \frac{\Delta t}{2} T_i^p + \frac{q'' \Delta r_f^2 \Delta t}{8} \quad \text{(Eq. D11)}
\]

\[
T_i^{p+1} - T_i^p = \frac{2 k_f \Delta t}{\rho c \Delta r^2} T_i^{p+1} - \frac{2 k_f \Delta t}{\rho c \Delta r^2} T_i^p + \frac{q'' \Delta t}{\rho c} \quad \text{(Eq. D12)}
\]
For the implicit solution, all $T^{p+1}$ variables on the LHS of the equation and all $T^p$ are on the RHS. After rearranging terms and applying the following definitions for $\alpha$ and $F_o$, Eq. D13 is the resulting finite difference equation for the left boundary.

\[
\alpha = \frac{k_f}{\rho c} \quad \quad F_o = \frac{\alpha \Delta t}{\Delta r^2} \\
(1 + 4F_o)T_i^{p+1} - 4F_o T_{i+1}^{p+1} = T_i^p + \frac{q'''' \Delta t}{\rho c}
\]  

(Eq. D13)

Derivation Finite Difference Equation: Fuel Interior Node

![Figure D-2: Interior Node](image)

Derivation of the finite difference equation for the interior node begins with Eq. D4. Integration is conducted in time from $t_p$ to $t_{p+1}$ and in space from $\left( r_i - \frac{\Delta r_f}{2} \right)$ to $\left( r_i + \frac{\Delta r_f}{2} \right)$.

\[
\int_{t_p}^{t_{p+1}} \int_{r_i-\frac{\Delta r_f}{2}}^{r_i+\frac{\Delta r_f}{2}} \frac{d}{dt} (\rho c T_r) \, dr \, dt = \int_{r_i-\frac{\Delta r_f}{2}}^{r_i+\frac{\Delta r_f}{2}} \left( k_f \frac{d}{dr} r \frac{dT}{dr} \right) \, dr \, dt + \int_{t_p}^{t_{p+1}} \int_{r_i-\frac{\Delta r_f}{2}}^{r_i+\frac{\Delta r_f}{2}} q'''' \, rdr \, dt
\]  

(Eq. D14)

Solving for the left side of Eq. D14 results in the following

\[
\int_{t_p}^{t_{p+1}} \int_{r_i-\frac{\Delta r_f}{2}}^{r_i+\frac{\Delta r_f}{2}} \frac{d}{dt} (\rho c T_r) \, dr \, dt = \int_{t_p}^{t_{p+1}} \rho c \frac{dT}{dt} \left[ \left( r_i + \frac{\Delta r_f}{2} \right)^2 - \left( r_i - \frac{\Delta r_f}{2} \right)^2 \right] \, dt\]  

(Eq. D15)

\[
\int_{t_p}^{t_{p+1}} \frac{\rho c}{2} \frac{dT}{dt} \left[ \left( r_i + \frac{\Delta r_f}{2} \right)^2 - \left( r_i - \frac{\Delta r_f}{2} \right)^2 \right] \, dt = \int_{t_p}^{t_{p+1}} \rho c \frac{dT}{dt} 2r_i \Delta r_f \, dt\]  

(Eq. D16)

\[
\int_{t_p}^{t_{p+1}} \frac{\rho c}{2} \frac{dT}{dt} 2r_i \Delta r_f \, dt = \rho c r_i \Delta r_f T_i^{p+1} - \rho c r_i \Delta r_f T_i^p
\]  

(Eq. D17)

When solving for the right side of Eq. D14, $t_{p+1} = t_p + \Delta t$. Solving for the right side of Eq. D14 results in the following

\[
\int_{t_p}^{t_{p+1}} \int_{r_i-\frac{\Delta r_f}{2}}^{r_i+\frac{\Delta r_f}{2}} \left( k_f \frac{d}{dr} r \frac{dT}{dr} \right) \, dr \, dt + \int_{t_p}^{t_{p+1}} \int_{r_i-\frac{\Delta r_f}{2}}^{r_i+\frac{\Delta r_f}{2}} q'''' \, rdr \, dt
\]  

(Eq. D18)
Combining Eq. D17 and Eq. D22 results in the following solution to the interior node finite difference equation.

\[ pcr_i \Delta r_i T_i^{p+1} - \rho c \eta \Delta r_i T_i^p = \left( \frac{k_f \Delta t}{\Delta r_f} \left( r_i + \frac{\Delta r_f}{2} \right) T_{i+1} + \frac{-k_f \Delta t}{\Delta r_f} \left( r_i - \frac{\Delta r_f}{2} \right) T_{i-1} \right) \Delta t + q'''r_i \Delta r_i \Delta t \]  

(Eq. D23)

Rearranging terms and defining constants in terms of \( \alpha \) and \( F_0 \) results in the following solution.

\[
-T_i^p + \frac{q''''' \Delta t}{\rho c} = \frac{k_f \Delta t}{\rho c \Delta r^2} \left( r_i + \frac{\Delta r_f}{2} \right) T_{i+1}^{p+1} + \frac{2k_f \Delta t}{\rho c \Delta r^2} T_i^{p+1} - \frac{k_f \Delta t}{\rho c \Delta r^2} \left( r_i - \frac{\Delta r_f}{2} \right) T_{i-1}^{p+1} \]

(Eq. D24)

\[
-T_i^p + \frac{q''''' \Delta t}{\rho c} = \frac{F_0}{\rho c} \left( r_i + \frac{\Delta r_f}{2} \right) T_{i+1}^{p+1} + \frac{F_0}{\rho c} \left( r_i + \frac{\Delta r_f}{2} \right) T_{i+1}^{p+1} = T_i^p + \frac{q''''' \Delta t}{\rho c} \]

(Eq. D25)
Derivation of the finite difference equation for the interface node between the fuel and gap begins with Eq. D4. Integration is conducted in time from \( t_p \) to \( t_{p+1} \). The integration in space is broken at the node \( r_i = r_f \). Thus integration over the fuel portion of the node is from \( r_f \) to \( r_f - \frac{\Delta r_f}{2} \) and over the gap portion of the node from \( r_f + \frac{\Delta r_g}{2} \) to \( r_f \). Additionally, only the fuel region of the nodal volume generates heat so there is only a \( q''' \) term associated with the integration from \( r_f \) to \( r_f - \frac{\Delta r_f}{2} \)

\[
\frac{d}{dt} (\rho c T r) = k_f \frac{d}{dr} r \frac{dr}{dr} + q''' r \tag{Eq. D26}
\]

The derivation begins by integrating over space and time the left side of Eq. 83

\[
\int_{t_p}^{t_{p+1}} \left[ \int_{r_f - \frac{\Delta r_f}{2}}^{r_f} \left( \rho_f c_f T r \right) dr + \int_{r_f}^{r_f + \frac{\Delta r_g}{2}} \left( \rho_g c_g T r \right) dr \right] dt \tag{Eq. D27}
\]

\[
\int_{t_p}^{t_{p+1}} \frac{dT}{dt} \left[ \frac{\rho_f c_f}{2} \left( r_f^2 - \left( r_f - \frac{\Delta r_f}{2} \right)^2 \right) + \frac{\rho_g c_g}{2} \left( \left( r_f + \frac{\Delta r_f}{2} \right)^2 - r_f^2 \right) \right] dt \tag{Eq. D28}
\]

Due to the complexity of this problem, a new term \( B_{fg} \) will be defined as

\[
B_{fg} = \frac{\rho_f c_f}{2} \left( r_f^2 - \left( r_f - \frac{\Delta r_f}{2} \right)^2 \right) + \frac{\rho_g c_g}{2} \left( \left( r_f + \frac{\Delta r_f}{2} \right)^2 - r_f^2 \right) \tag{Eq. D29}
\]

\[
\int_{t_p}^{t_{p+1}} \frac{dT}{dt} B_{fg} dt = B_{fg} T_{i}^{p+1} - B_{fg} T_{i}^{p} \tag{Eq. D30}
\]

Next one must integrate the right side of Eq. D30 over space and time. Recall that \( t_p \) to \( t_{p+1} \) is equal to \( t + \Delta t \)
\begin{align*}
\int_t^{t+\Delta t} & \left[ \int_{r_f-\frac{\Delta r_f}{2}}^{r_f} k_f \frac{d}{dr} r \frac{dT}{dr} dr + \int_{r_f-\Delta r_f}^{r_f} \frac{q''r^2}{2} \frac{dr}{dr} r \frac{dT}{dr} dr + \int_{r_f}^{r_f+\frac{\Delta r_f}{2}} k_g \frac{d}{dr} r \frac{dT}{dr} dr \right] dt \quad \text{(Eq. D31)} \\
\int_t^{t+\Delta t} & \left[ k_f \frac{\Delta r_f}{2} \frac{dT}{dr} \bigg|_{r_f} + k_f \left( r_f - \frac{\Delta r_f}{2} \right) \frac{dT}{dr} \bigg|_{r_f-\Delta r_f} + \frac{q''}{2} \left( r_f^2 - \left( r_f - \frac{\Delta r_f}{2} \right)^2 \right) \right] dt \\
\quad + k_g \left( r_f + \frac{\Delta r_g}{2} \right) \frac{dT}{dr} \bigg|_{r_f+\frac{\Delta r_g}{2}} - k_g r_f \frac{dT}{dr} \bigg|_{r_f} \quad \text{(Eq. D32)} \\
\text{BC: } q'' f & = q'' g \rightarrow k_f r_f \frac{dT}{dr} \bigg|_{r_f} = k_g r_f + \frac{dT}{dr} \bigg|_{r_f} \\
\left[ -k_f \left( r_f - \frac{\Delta r_f}{2} \right) \left( \frac{T_i-T_{i-1}}{\Delta r_f} \right) + \frac{q''}{2} \left( r_f^2 - \left( r_f - \frac{\Delta r_f}{2} \right)^2 \right) + k_g \left( r_f + \frac{\Delta r_g}{2} \right) \left( \frac{T_{i+1}-T_i}{\Delta r_g} \right) \right] \Delta t \quad \text{(Eq. D33)}
\end{align*}

Combining Eq. D30 and Eq. D33 results in the following:

\begin{align*}
B_{fg} T_i^{p+1} - B_{fg} T_i^p & = -\frac{k_f}{\Delta r_f} \left( r_f - \frac{\Delta r_f}{2} \right) T_i^{p+1} + \frac{k_f}{\Delta r_f} \left( r_f - \frac{\Delta r_f}{2} \right) T_{i-1}^{p+1} + \frac{q''}{2} \left( r_f^2 - \left( r_f - \frac{\Delta r_f}{2} \right)^2 \right) + \frac{k_g}{\Delta r_g} \left( r_f + \frac{\Delta r_g}{2} \right) T_{i+1}^{p+1} - \frac{k_g}{\Delta r_g} \left( r_f + \frac{\Delta r_g}{2} \right) T_i^{p+1} \\
\frac{1}{\partial T_i} & \left[ -\frac{k_f}{\Delta r_f} \left( r_f - \frac{\Delta r_f}{2} \right) T_i^{p+1} + \left( B_{fg} + \frac{k_f}{\Delta r_f} \left( r_f - \frac{\Delta r_f}{2} \right) + \frac{k_g}{\Delta r_g} \left( r_f + \frac{\Delta r_g}{2} \right) \right) T_{i+1}^{p+1} \right] = T_i^p + \frac{q''}{2 B_{fg}} \left( r_f^2 - \left( r_f - \frac{\Delta r_f}{2} \right)^2 \right) \Delta t \quad \text{(Eq. D34)}
\end{align*}

\begin{align*}
B_{fg} & = \rho_f c_f \left( r_f^2 - \left( r_f - \frac{\Delta r_f}{2} \right)^2 \right) + \rho_g c_g \left( r_f + \frac{\Delta r_g}{2} \right)^2 \quad \text{(Eq. D35)}
\end{align*}
Derivation of the finite difference equation for the interior node begins with Eq. D4. Integration is conducted in time from \( t_p \) to \( t_{p+1} \) and in space from \( r_i \) to \( r_i + \frac{\Delta r_g}{2} \). This derivation is the same process for the cladding material however the constants will change. No heat is generated in the gap or the cladding so \( q''' = 0 \). Thus, Eq. 61 becomes

\[
\frac{d}{dt}(pcT_i) = k_g \frac{d}{dr} r \frac{dT_i}{dr}
\]  
(Eq. D36)

The derivation begins by integrating the left side of Eq. 93 over space and time.

\[
\int_{t_p}^{t_{p+1}} \int_{r_i - \frac{\Delta r_g}{2}}^{r_i + \frac{\Delta r_g}{2}} \rho_g c_g T_i \frac{dT_i}{dr} dr dt = \int_{t_p}^{t_{p+1}} \frac{\rho_g c_g}{2} T_i \left( \left( r_i + \frac{\Delta r_g}{2} \right)^2 - \left( r_i - \frac{\Delta r_g}{2} \right)^2 \right) dt
\]  
(Eq. D37)

Next integrate the right side of Eq. D37 over space and time.

\[
\frac{k_g \Delta T_i}{\Delta r_g} (r_i + \frac{\Delta r_g}{2}) T^{p+1}_{i+1} - \frac{k_g \Delta T_i}{\Delta r_g} (r_i - \frac{\Delta r_g}{2}) T^{p+1}_i - \frac{k_g \Delta T_i}{\Delta r_g} T^{p+1}_{i-1} + \frac{k_g \Delta T_i}{\Delta r_g} (r_i + \frac{\Delta r_g}{2}) T^{p+1}_{i+1}
\]  
(Eq. D39)

Combining Eq. D38 and D39 results in the following

\[
\rho_g c_g r_i \Delta r_g T^{p+1}_i - \rho_g c_g r_i \Delta r_g T^{p}_i = \frac{k_g \Delta T_i}{\Delta r_g} (r_i + \frac{\Delta r_g}{2}) T^{p+1}_{i-1} - \frac{k_g \Delta T_i}{\Delta r_g} T^{p+1}_i - \frac{k_g \Delta T_i}{\Delta r_g} (r_i - \frac{\Delta r_g}{2}) T^{p+1}_i + \frac{k_g \Delta T_i}{\Delta r_g} (r_i + \frac{\Delta r_g}{2}) T^{p+1}_{i+1}
\]  
(Eq. D41)

\[
-F_0 \frac{r_i}{r_i} (r_i - \frac{\Delta r_g}{2}) T^{p+1}_{i-1} + (1 + 2F_0) T^{p+1}_i - \frac{r_i}{r_i} (r_i + \frac{\Delta r_g}{2}) T^{p+1}_{i+1} = T^{p}_i
\]  
(Eq. D42)
Derivation of the finite difference equation for the interface node between the gap and cladding begins with Eq. D4. Integration is conducted in time from \( t_p \) to \( t_{p+1} \). The integration in space is broken at the node \( r_i \). Thus integration over the gap portion of the node is from \( r_i \) to \( (r_i - \frac{\Delta r_g}{2}) \) and over the cladding portion of the node from \( (r_i + \frac{\Delta r_c}{2}) \) to \( r_i \). Additionally, neither the gap region nor the cladding region generates heat. Therefore there is no \( q''' \) term for this problem.

\[
\frac{d}{dt} \left( \rho c T_r \right) = k \frac{dr}{dr} \frac{dT}{dr}
\]  
(Eq. D43)

The derivation begins with integrating the left side of Eq. 100 over space and time.

\[
\int_{t_p}^{t_{p+1}} \left[ \int_{r_i}^{r_i+\Delta r_g} \left( \rho c g Tr \right) dr + \int_{r_i}^{r_i+\Delta r_c} \left( \rho c c Tr \right) dr \right] dt 
\]  
(Eq. D44)

\[
\int_{t_p}^{t_{p+1}} \frac{dT}{dt} \left[ \frac{\rho g c g}{2} \left( r_i^2 - \left( r_i - \frac{\Delta r_g}{2} \right)^2 \right) \right] dt
\]  
(Eq. D45)

Due to the complexity of this term, a new term \( B_{gc} \) is defined as

\[
B_{gc} = \frac{\rho g c g}{2} \left( r_i^2 - \left( r_i - \frac{\Delta r_g}{2} \right)^2 \right) + \frac{\rho c c c}{2} \left( \left( r_i + \frac{\Delta r_c}{2} \right)^2 - r_i^2 \right)
\]  
(Eq. D46)

\[
\int_{t_p}^{t_{p+1}} \frac{dT}{dt} dt = B_{gc} T_i^{p+1} - B_{gc} T_i^p
\]  
(Eq. D47)

The next step is to integrate the ride side of Eq. D44 over space and time.

\[
\int_{t}^{t+\Delta t} \left[ \int_{r_i-\frac{\Delta r_g}{2}}^{r_i} k_g \frac{dr}{dr} \frac{dT}{dr} dr + \int_{r_i}^{r_i+\frac{\Delta r_c}{2}} k_c \frac{dr}{dr} \frac{dT}{dr} dr \right] dt
\]  
(Eq. D48)
\[ \int_t^{t+\Delta t} \left[ k_g r_i \frac{dT}{dr} \bigg|_{r_i} - k_g \left( r_i - \frac{\Delta r_g}{2} \right) \frac{dT}{dr} \bigg|_{r_i - \frac{\Delta r_g}{2}} + k_c \left( r_i + \frac{\Delta r_c}{2} \right) \frac{dT}{dr} \bigg|_{r_i + \frac{\Delta r_c}{2}} - k_c r_i \frac{dT}{dr} \bigg|_{r_i} \right] dt \quad \text{(Eq. D49)} \]

\[ q''_g = q''_c \rightarrow k_g r_i \frac{dT}{dr} \bigg|_{r_i} = k_c r_i \frac{dT}{dr} \bigg|_{r_i} \]

\[ \left[ -k_g \left( r_i - \frac{\Delta r_g}{2} \right) \left( \frac{T_{i+1}-T_i}{\Delta r_g} \right) + k_c \left( r_i + \frac{\Delta r_c}{2} \right) \left( \frac{T_{i+1}-T_i}{\Delta r_c} \right) \right] \Delta t \quad \text{(Eq. D50)} \]

Combine Eq. D47 and D50 to get the following

\[ B_{ge} T_i^{p+1} - B_{ge} T_i^p = \frac{k_g \Delta t}{\Delta r_g} \left( r_i - \frac{\Delta r_g}{2} \right) T_i^{p+1} - \frac{k_g \Delta t}{\Delta r_g} \left( r_i - \frac{\Delta r_g}{2} \right) T_i^{p+1} + \frac{k_c \Delta t}{\Delta r_c} \left( r_i + \frac{\Delta r_c}{2} \right) T_i^{p+1} - \frac{k_c \Delta t}{\Delta r_c} \left( r_i + \frac{\Delta r_c}{2} \right) T_i^{p+1} \quad \text{(Eq. D51)} \]

\[ \frac{1}{B_{ge}} \left[ -\frac{k_g \Delta t}{\Delta r_g} \left( r_i - \frac{\Delta r_g}{2} \right) T_i^{p+1} + \left( B_{ge} + \frac{k_g \Delta t}{\Delta r_g} \left( r_i - \frac{\Delta r_g}{2} \right) + \frac{k_c \Delta t}{\Delta r_c} \left( r_i + \frac{\Delta r_c}{2} \right) \right) T_i^{p+1} - \frac{k_c \Delta t}{\Delta r_c} \left( r_i + \frac{\Delta r_c}{2} \right) T_i^{p+1} \right] = T_i^{p+1} \quad \text{(Eq. D52)} \]

\[ B_{ge} = \frac{\rho_g c_p}{2} \left( r_i^2 - \left( r_i - \frac{\Delta r_g}{2} \right)^2 \right) + \frac{\rho_c c_p}{2} \left( \left( r_i + \frac{\Delta r_c}{2} \right)^2 - r_i^2 \right) \]
Derivation Finite Difference Equation: RHS Node (Exterior)

[Diagram of exterior node with labels and equations]

Derivation of the finite difference equation for the right most node begins with Eq. D4. Integration is performed over time from \( t_p \) to \( t_{p+1} \) and over space from the boundary, \( r=R \) to \( \left(r_i - \frac{\Delta r_c}{2}\right) \). No heat is generated in the cladding thus \( q'''=0 \). Therefore, Eq. D4 takes on the following form.

\[
\frac{d}{dt}(\rho c T r) = k_c \frac{d}{dr} r \frac{dT}{dr}
\]  \hspace{1cm} \text{(Eq. D53)}

The derivation begins by integrating the left side of Eq. D53 over space and time.

\[
\int_{t_p}^{t_{p+1}} \int_{r_i - \frac{\Delta r_c}{2}}^{R} \left( \frac{d}{dt}(\rho c c T r) \right) dr dt = \int_{t_p}^{t_{p+1}} \frac{\rho c c T}{2} \left(R^2 - \left(r_i - \frac{\Delta r_c}{2}\right)^2\right) dt
\]  \hspace{1cm} \text{(Eq. D54)}

Due to the complexity of the problem, a new term, \( B_c \), is defined as

\[
B_c = \frac{\rho c c}{2} \left(R^2 - \left(r_i - \frac{\Delta r_c}{2}\right)^2\right)
\]  \hspace{1cm} \text{(Eq. D55)}

\[
\int_{t_p}^{t_{p+1}} \frac{dT}{dt} B_c dt = B_c T_i^{p+1} - B_c T_i^p
\]  \hspace{1cm} \text{(Eq. D56)}

The next step is to integrate the right side of Eq. D53 over space and time.

\[
\int_{t}^{t+\Delta t} \left( k_c R \frac{dT}{dr} \bigg|_R - k_c \left(r_i - \frac{\Delta r_c}{2}\right) \frac{dT}{dr} \right) dt
\]  \hspace{1cm} \text{(Eq. D57)}

Recall the following boundary condition

\[
-k_c \frac{dT}{dr} \bigg|_R = h_\infty (T_R - T_\infty)
\]

\[
\left[-(h_\infty (T_R - T_\infty)) - k_c \left(r_i - \frac{\Delta r_c}{2}\right) \left(\frac{T_R - T_i}{\Delta r_c}\right)\right] \Delta t
\]

\[
-h_\infty \Delta t T_R^{p+1} + h_\infty \Delta t T_i + k_c \frac{\Delta t}{\Delta r_c} \left(r_i - \frac{\Delta r_c}{2}\right) T_R^{p+1} + k_c \frac{\Delta t}{\Delta r_c} \left(r_i - \frac{\Delta r_c}{2}\right) T_i^{p+1}
\]  \hspace{1cm} \text{(Eq. D58)}

Combine Eq. D56 and Eq. D59 to get the following
\[ B_c T_i^{p+1} - B_c T_i^p = - \left[ \frac{h_\infty \Delta t}{k_c \Delta r_c} \left( r_i - \frac{\Delta r_c}{2} \right) \right] T_R^{p+1} + \frac{k_c \Delta t}{\Delta r_c} \left( \eta_i - \frac{\Delta r_c}{2} \right) T_i^{p+1} + h_\infty \Delta t T_\infty \]  

(Eq. D60)

\[ \frac{1}{B_c} \left[ \left( B_c + h_\infty \Delta t + \frac{k_c \Delta t}{\Delta r_c} \left( r_i - \frac{\Delta r_c}{2} \right) \right) T_R^{p+1} \right] = T_i^p + \frac{h_\infty \Delta t T_\infty}{B_c} \]  

(Eq. D61)

\[ B_c = \frac{\rho_c c_c}{2} \left( R^2 - \left( \eta_i - \frac{\Delta r_c}{2} \right)^2 \right) \]
**Implicit Solution Set-Up**

Generating a solution for temperature at each node after \( p \) iterations can only be done using a matrix solution method. The matrix solution will assume the following form

\[
\bar{A}T^{p+1} = \bar{T}^p + \bar{C}
\]

To solve for \( T^{p+1} \), the solution assumes an iterative form. The solution begins with a guess of \( T^p \) that is equal to the solution of the steady-state equation. One then finds \( T^{p+1} \) using the following approach.

\[
T^{p+1} = \bar{A}^{-1}T^p + \bar{A}^{-1}\bar{C}
\]

The \( T^{p+1} \) will become the new \( T^p \) and the iteration will continue for the number of \( p \) time steps.
Appendix D: MATLAB Code

The Matlab code developed and used for this analysis is found in the zip file “Group1_FinalProject_code.zip,” which will be uploaded separately.